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ON THE TABULATION OF INDEFINITE INTEGRALS.

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Abstract.

A number of efficient methods have been developed for the
numerical solution of linear second order differential
equations from which the term involving the first deriva-
tive is absent. The function $\theta(z) \int_a^{g(z)} \phi(t)dt + h(z)$ satis-
fies such an equation. This fact is of material assistance
in the tabulation of indefinite integrals.

The tabulation of an auxiliary function which is related
to an indefinite integral of the form $\int_\alpha^x \phi(t)dt$ may often
be most economically carried out by the numerical integra-
tion of a differential equation which is satisfied by the
auxiliary function. The choice of auxiliary function and
auxiliary variable z where $x = g(z)$ is in the main
determined by facilities for interpolation in the result-
ing table, and is logically that which provides the great-
est information about the required indefinite integral in
the smallest possible space. Of necessity the auxiliary
function must be simply related to the indefinite integral,
so that the latter may easily be extracted from the former,
the final choice of auxiliary function and auxiliary vari-
able being resolved from the conflict of economy of ex-
pression on the one hand and ease of definition on the
other. In general it is true to say that the considerations
which lead to the choice of auxiliary function and variable
are precisely those which make the numerical solution of
the differential equation which the function satisfies most

pleasant. If an auxiliary function has been chosen which
is easy to interpolate, then the appropriate differential
equation is in general correspondingly easy to integrate
numerically, for the implication of the first statement
is that the successive differences of the function decrease
smoothly, which is a condition for the second. A form of
differential equation which particularly lends itself to
numerical integration[1] is that which is linear and of the
second order and in which the term in the first derivative
is absent. In view of this fact it is of interest to de-
rive the following elementary

THEOREM

If $y(z) = \Theta(z) \int_{\alpha}^{g(z)} \phi(t)dt + h(z)$ (1)

then $y(z)$ satisfies a linear differential equation of the
second order in which the term involving $y'(z)$ is absent.
For, from (1)

$$y'(z) = \Theta'(z) \int_{\alpha}^{g(z)} \phi(t)dt + \Theta(z)g'(z)\phi(g(z)) + h'(z)$$

and hence $\Theta(z)y'(z) - \Theta'(z)y(z) = \Theta^2(z)g'(z)\phi(g(z)) +$

$$+ \Theta(z)h'(z) - \Theta'(z)h(z)$$

and by differentiating again

$$\Theta(z)y''(z) - \Theta''(z)y(z) = \left[\Theta^2(z)g'(z)\phi(g(z))\right]' +$$

$$\Theta(z)h''(z) - \Theta''(z)h(z)$$ (2)

where primes denote differentiation with respect to z.

Since the computationally unpleasant behaviour of the in-

definite integral $\int_{\alpha}^{z} \phi(t)dt$ may well arise from the be-

haviour of $\phi(x)$, one obvious choice for $\Theta(z)$ is

$$\Theta(z) = \left[\phi(g(z))\right]^{-1}$$

In this case with $h(z) = 0$ and $x = g(z) = z$ equation
(2) becomes

$$\Theta(z)y''(z) - \Theta''(z)y(z) = \Theta'(z)$$ (3)

Another choice which would apply when $\phi(x)$ is dominated by
a term of the form x^n, i.e. when

$$\phi(x) = kx^n + \epsilon(x)$$

where $\epsilon(x)$ is relatively small throughout the range of integration, is

$$\theta(z) = \left[x\phi(x) - \alpha\phi(\alpha)\right]^{-1} \qquad (4)$$

For if $\phi(x) = kx^n$, then $y(z)$ is constant and little difficulty is to be expected in its tabulation, whilst if $\phi(x) \neq kx^n$ the substitution (4) is at least a reasonable step which can be made in the hope of easing tabulation. In this case, when $h(z) = 0$ it is necessary to eliminate $v, g(z)$ and $\phi(x)$ from

$$x = g(z) = z \; ; \;\; \theta(z)v''(z) - \theta''(z)v(z) = \left[\theta^2(z)g'(z)\phi(g(z))\right]'$$

and equation (4). Thus the required version of equation (2) is

$$\theta(z)y''(z) - \theta''(z)v(z) = \left[\left(1 + \alpha\phi(\alpha)\theta(z)\right)\theta(z)/z\right]' \qquad (5)$$

For large values of the argument the simplest auxiliary variable to use is $z = {}^1/x$, i.e. $x = g(z) = {}^1/z$. In this case, with $h(z) = 0$ and $\theta(z) = \left[\phi(g(z))\right]^{-1}$, equation (2) becomes

$$\theta(z)v''(z) - \theta''(z)v(z) = -\left[\theta(z)/z^2\right]'$$

vis à vis equation (3); and with $h(z) = 0$, $\theta(z)$ again being given by equation (4), equation (2) becomes

$$\theta(z)y''(z) - \theta''(z)v(z) = -\left[\left(1 + \alpha\phi(\alpha)\theta(z)\right)\theta(z)/z\right]'$$

vis à vis equation (5).

An example in which the above theorem may be applied can be taken from the literature. Fox and Miller[2] tabulated the function $T(z)$ where $T(z) = e^{-x}Ei(x) - x^{-1}$, $z = x^{-1}$ and $Ei(x) = \int_{-\infty}^{x} t^{-1}e^{t}dt$, for $z = -0 \cdot 1(0 \cdot 01)0 \cdot 1$, by integrating the equation

$$T'' - (z^{-4} - 2z^{-3})T + z^{-2} = 0$$

What seems to be a slightly better choice of auxiliary function was adopted by Vickers[3] who tabulates $S(z)$ over the same range where

$$S(z) = xe^{-x}Ei(x)$$

(This corresponds in the general case to $\theta(z) = [\phi(x)]^{-1}$
and $S(z)$ satisfies the differential equation,

$$z^4 S'' - (2z^2 - 4z + 1)S = 3z - 1$$

though Vickers did not integrate this equation to obtain
his values). Results displayed in reference 2 indicate
that facilities for interpolation in $S(z)$ are better
than for those in $T(z)$.

The mathematics in the above work is of a somewhat naive
order but it is felt that a statement of the general case
serves to indicate one possible line of attack in the ta-
bulation of indefinite integrals, and may save a little
of the time of someone who is engaged upon this problem.

I am indebted to Miss D.B. Catton for useful discussion of
the above work.

References.

[1]. D.R. HARTREE, "Numerical Analysis", Oxford University
                  Press, 1952, Ch. VII, p.126.

[2]. FOX L. and MILLER J.C.P., "Table Making for Large Ar-
                  guments. The Exponential Integral".
                  M.T.A.C. V.No.35, July 1951, p.163-167.

[3]. FOX L. and MILLER J.C.P., loc.cit., p.166.

ALGOL Programming.

CONTRIBUTION NO. 2.
SIMPSON NUMERICAL INTEGRATION WITH VARIABLE LENGTH OF STEP.
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```
procedure  simpson(a,b,h,ut,lt,fct,sum,g);
           value a,b,h,ut,lt;   real a,b,h,ut,lt,sum,g;
           real procedure fct;
comment    simpson computes the approximate value of the
           definite integral of fct(x) from a to b with
           automatic search for the appropriate length of
           step. Input parameters are: a and b, the lower
           and upper boundaries of the quadrature interval
           (a < b), h the proposed length of step between
           used function values, ut and lt, the upper and
           lower tolerance factors, fct the function. Out-
           put parameters are: sum the value of the inte-
           gral, g the average length of step. The proce-
           dure will usually give better accuracy than it
           is asked for but should nevertheless be used
           with discrimination;
begin      integer i; real x,F1,F2,F3,F4,F5,eps,p,q;
           Boolean ready ;
           i:=0 ; sum:=0 ; ready:=false; g:=h ; x:=a ;
           F1:=fct(x);
loopstart: i:=i+1 ; if x + 4×g ≥ b then begin g:=(b-x)/4;
           ready:=true end;
           F5:=fct(x+4×g); F3:=fct(x+2×g);
innerloop: F2:=fct(x+g); F4:=fct(x+3×g);
           p:= F1 + 4×F3 + F5; q:=p - 2×F3 + 4×(F2+F4);
           eps:= (2×p-q)×g/45 ;
           if abs(q×g×ut/3) > abs(eps) then go to nottobad;
           ready:=false ; g:=g/2 ; F5:=F3; F3:=F2;
           go to innerloop;
nottobad:  sum:=sum + q×g/3 - eps ;
           if ready then go to final ;
           x:=x + 4×g ;
           if abs(q×g×lt/3) ≥ abs(eps) then g:=2×g;
           F1:=F5 ; go to loopstart;
final:     g:= (b-a)/i/4

end;
```

The Program has been successfully tested using the ALGOL Compiler of
FACIT Electronics, Gothenburg.